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A linear model of intermediate statistics

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Abstract. Approximate Hamiltonians for the one-dimensional (1D) Calogero and two-dimensional (2D) anyon models in a harmonic well are constructed. These Hamiltonians are exactly diagonalizable, and their spectra interpolate linearly between the Bose statistics and the Fermi statistics. In particular, in 2D, the thermodynamics is similar to that of a system obeying a generalized exclusion principle and may be viewed as a starting approximation for the thermodynamics of anyons.

Many generalizations of the concept of quantum statistics have been considered in recent decades [1]. The 1D Calogero model and the 2D anyon model are examples. In these two models, the particles interpolate between bosons and fermions or more generally between the conjugate representations of the symmetric group when the statistical coupling ν goes from 0 to 1 [2, 3]. Albeit the Haldane generalization of the Pauli principle is satisfied in some particular cases, as the Calogero model in the Bose and Fermi representations [4], or anyons in the lowest Landau level [5], the general situation is often more complicated [6, 7].

In this paper, I focus on the remarkable properties of the solution of the Calogero model to first order in perturbation theory. In this way, I define a 'linear model' of intermediate statistics in 1D, the particles of which are in a harmonic well and only interact by interchanges. This model is exactly solved for all exchange symmetries of the wavefunction. Its spectrum interpolates linearly between the conjugate representations of S_N and of course involves all the linear energies of the Calogero model. A consistent thermodynamics is obtained in the thermodynamic limit. The linear model of intermediate statistics is immediately generalizable in 2D where it appears intimately connected to the anyon model. Albeit its eigenfunctions are not altered by the interaction, the linear model produces a thermodynamics similar to that of a system obeying a generalized exclusion principle. In particular, it may be viewed as a starting approximation for the thermodynamics of anyons.

Here, one considers a quantum system of N particles with a Hamiltonian H. p is a permutation of particles, P is its conjugate class, Y is a Young pattern or a Young projector [8]. The class P is identified with the partition $[\{L^{\nu_L}\}]$ of N where ν_L is the number of cycles of length L in $p \in P$. For convenience, a Young pattern will be denoted by a partition between round brackets, namely $(\{\lambda_l\})$ where λ_l is the number of cases in the lth line.

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One defines the partition function $Z_Y = \mathrm{tr}\,\mathrm{e}^{-\beta H} Y$ and the class function $Z_P = \mathrm{tr}\,\mathrm{e}^{-\beta H} p$, with $\beta^{-1} = k_\mathrm{B}T$. The latter trace is over the whole Hilbert space, it may be expanded with respect to irreducible representations as

$$Z_P = \sum_Y \chi_Y(P) Z_Y \tag{1}$$

where the character $\chi_Y(P)$ [9] originates from the trace of p in each irreducible state space generated by the permutations of a state $Y\psi \neq 0$. The inverse relation

$$Z_Y = \sum_P \frac{d_P}{N!} \chi_Y(P) Z_P \tag{2}$$

is obtained by using the orthogonality relations for characters. $d_P = N!/\prod_L \nu_L! L^{\nu_L}$ is the number of permutations of class P.

In the thermodynamic limit, the thermodynamics is determined by the connected parts Z_Y^c and Z_P^c, which are additive quantities. In particular, $Z^\mathrm{c}_{(N)}$, $Z^\mathrm{c}_{(1^N)}$ and $(1/N!)Z^\mathrm{c}_{[1^N]}$ reproduce the cluster coefficients b_N in Bose, Fermi and Boltzmann statistics, respectively [10]. The connected parts of the class functions can be defined in a standard way by

$$\begin{aligned}
&Z_{[N]} = Z^\mathrm{c}_{[N]}\\
&Z_{[L_1,L_2]} = Z^\mathrm{c}_{[L_1,L_2]} + Z^\mathrm{c}_{[L_1]}Z^\mathrm{c}_{[L_2]}\\
&Z_{[L_1,L_2,L_3]} = Z^\mathrm{c}_{[L_1,L_2,L_3]} + Z^\mathrm{c}_{[L_1,L_2]}Z^\mathrm{c}_{[L_3]} + Z^\mathrm{c}_{[L_1,L_3]}Z^\mathrm{c}_{[L_2]} + Z^\mathrm{c}_{[L_2,L_3]}Z^\mathrm{c}_{[L_1]} + Z^\mathrm{c}_{[L_1]}Z^\mathrm{c}_{[L_2]}Z^\mathrm{c}_{[L_3]}
\end{aligned} \tag{3}$$

etc. We can solve these equations successively and obtain

$$\begin{aligned}
&Z^\mathrm{c}_{[N]} = Z_{[N]}\\
&Z^\mathrm{c}_{[L_1,L_2]} = Z_{[L_1,L_2]} - Z_{[L_1]}Z_{[L_2]}\\
&Z^\mathrm{c}_{[L_1,L_2,L_3]} = Z_{[L_1,L_2,L_3]} - Z_{[L_1,L_2]}Z_{[L_3]} - Z_{[L_1,L_3]}Z_{[L_2]} - Z_{[L_2,L_3]}Z_{[L_1]} + 2Z_{[L_1]}Z_{[L_2]}Z_{[L_3]}
\end{aligned} \tag{4}$$

etc. In a perturbative expansion [3], the connected part reduces to the series of connected diagrams. The relations (1) and (2) apply equally well to the connected parts Z_Y^c and Z_P^c, and will be useful to compute b_N in Bose and Fermi statistics.

Note that the particles will be confined in a harmonic well of frequency ω rather than in a box of volume V. The $\omega \to 0$ limit can be identified with the thermodynamic limit if the divergent factor $N^{-d/2}(\beta\omega)^{-d}$ surviving in the connected parts is identified with an additive factor, namely $V\lambda_T^{-d}$ where d is the space dimension and $\lambda_T = \hbar\sqrt{2\pi\beta/m}$ is the thermal wavelength [11].

Let me first review some aspects of the system of N identical harmonic oscillators. The Hamiltonian is

$$H = \sum_{i=1}^N \left(-\tfrac{1}{2}\partial_i^2 + \tfrac{1}{2}\omega^2 x_i^2\right) \tag{5}$$

in units $\hbar = 1$, $m = 1$. The creation (annihilation) operators are

$$a_i^\pm = \sqrt{\frac{\omega}{2}}x_i \mp \frac{\partial_i}{\sqrt{2\omega}} \qquad [H, a_i^\pm] = \pm\omega a_i^\pm. \tag{6}$$

A basis in the ring of the symmetric polynomials in the N commuting variables $a_i^\pm$ is given by

$$A_k^\pm = \sum_{i=1}^N (a_i^\pm)^k \qquad [H, A_k^\pm] = \pm k\omega A_k^\pm \tag{7}$$

with $k = 1, 2, \ldots, N$. These operators constitute a complete set of raising (lowering) operators preserving the exchange symmetry of the wavefunction. For a definite exchange symmetry, there is one ground state ψ_0 annihilated by the A_k's and a set of excited states

$$\psi = \prod_{k=1}^{N} (A_k^+)^{n_k} \psi_0 \tag{8}$$

with the energies

$$E = \sum_{k=1}^{N} k n_k \omega + E_0 \tag{9}$$

where the n_k's are non-negative integers. According to permutation theory, there are $N!$ independent ground states: one in the Bose representation, another in the Fermi representation, and d_Y degenerate ground states in each irreducible representation of dimension d_Y of S_N. The ground states may be deduced from the action of some non-symmetric homogeneous polynomials in the a_i^+'s onto the bosonic ground state $\exp(-\frac{1}{2}\omega \sum x_i^2)$. Thus, the energy of a ground state is $E_0 = \frac{1}{2}N\omega + d\omega$ if d is the degree of the homogeneous polynomial. In particular, the fermionic ground state follows from the action of the antisymmetric operator $\prod_{i,j<}(a_i^+ - a_j^+)$ and its energy is $\frac{1}{2}N\omega + \frac{1}{2}N(N-1)\omega$. The general case is outlined in [3]. For instance, one finds the bases $\{a_1^+ - a_2^+, a_1^+ - a_3^+\}$ and $\{(a_1^+ - a_2^+)(a_1^+ + a_2^+ - 2a_3^+), (a_1^+ - a_3^+)(a_1^+ + a_3^+ - 2a_2^+)\}$ for the two equivalent representations of dimension 2 of S_3.

In this approach, the complete set of quantum numbers is n_k, Y, a and b where the n_k's are the preceding raising numbers, Y is a Young pattern of S_N, $a = 1, 2, \ldots, d_Y$ labels the equivalent irreducible representations associated with Y and $b = 1, 2, \ldots, d_Y$ labels the elements of a ground-state basis for the representation of indexes Y and a. Since all the ground states of an irreducible representation are combinations of the permutations of one of them, their energies are identical, that is

$$E_0 = \tfrac{1}{2}N\omega + d_{Y,a}\omega \tag{10}$$

does not depend on b. One can select one of these degenerate ground states with a Young projector Y, as in the definition of the partition function Z_Y.

Performing the trace over the n_k's, one gets the partition function in terms of the ground-state energies,

$$Z_Y = \sum_a \frac{\exp[-(d_{Y,a} - N(N-1)/4)\beta\omega]}{\prod_{k=1}^{N} 2\mathrm{sh}(k\beta\omega/2)}. \tag{11}$$

The energy (10) is $\frac{1}{2}N\omega$ for the Bose ground state and $\frac{1}{2}N\omega + \frac{1}{2}N(N-1)\omega$ for the Fermi one. In another representation, the ground-state energies may be obtained by identifying (11) with (2) where

$$Z_P = \prod_L \left(\frac{1}{2\mathrm{sh}(L\beta\omega/2)} \right)^{\nu_L} \tag{12}$$

is first rewritten as a sum of certain terms of the expansion (11) by means of algebraic operations—this is always possible due to (1). In this way, one finds $\{d_{(2),a}\} = \{0\}$, $\{d_{(11),a}\} = \{1\}$, $\{d_{(3),a}\} = \{0\}$, $\{d_{(21),a}\} = \{1, 2\}$, $\{d_{(111),a}\} = \{3\}$, $\{d_{(4),a}\} = \{0\}$, $\{d_{(31),a}\} = \{1, 2, 3\}$, $\{d_{(22),a}\} = \{2, 4\}$, $\{d_{(211),a}\} = \{3, 4, 5\}$, $\{d_{(1111),a}\} = \{6\}$; etc. Note the symmetry $d_{Y,a} = \frac{1}{2}N(N-1) - d_{\bar{Y},1+d_Y-a}$ between conjugate patterns Y and $\bar{Y}$.

Let me now consider a system of identical harmonic oscillators interacting by an attractive Calogero potential, that is

$$H = \sum_{i=1}^{N} \left(-\tfrac{1}{2}\partial_i^2 + \tfrac{1}{2}\omega^2 x_i^2\right) + \sum_{i,j<} \frac{\nu(\nu-1)}{(x_i - x_j)^2} \tag{13}$$

for $\nu \in [0, 1]$. This model is rather different from the Calogero one for a repulsive potential. In fact, it is a 1D model of intermediate statistics analogous to the 2D anyon model [2]. Suppose that one wishes to discuss the interaction to first order in perturbation theory. To do this, it is advisable to do the transformation $\psi = \Pi_{i,j<}|x_i - x_j|^{\nu}\tilde{\psi}$ in order to work with a Hamiltonian

$$\tilde{H} = \sum_{i=1}^{N} \left(-\tfrac{1}{2}\partial_i^2 + \tfrac{1}{2}\omega^2 x_i^2\right) - \sum_{i,j<} \frac{\nu}{x_i - x_j}(\partial_i - \partial_j) \tag{14}$$

whose matrix elements are well defined with the principal value regularization. Then, it suffices to diagonalize the restriction of the perturbation inside each unperturbed degenerate subspace. The restriction of the two-body interaction $-\nu x_{ij}^{-1}\partial_{ij}$ is necessarily $\nu\omega p_{ij}$, where p_{ij} represents the transposition of x_i and x_j, because this restriction must reproduce the linear slopes $\pm\nu\omega$ of the two-body spectrum if the wavefunction is symmetric (antisymmetric) with respect to x_i, x_j.

From now on, we focus on the Hamiltonian

$$H = \sum_{i=1}^{N} \left(-\tfrac{1}{2}\partial_i^2 + \tfrac{1}{2}\omega^2 x_i^2\right) + \sum_{i,j<} \nu\omega p_{ij}. \tag{15}$$

Although this Hamiltonian is defined above as a first-order approximation for the Calogero model in a harmonic well, we will see that it leads to consistent and interesting results in the non-perturbative domain $\nu \in]-\infty, \infty[$. First, note that $\sum_{i,j<} p_{ij}$ is known as the transposition class operator in group theory [8]. Any state of an irreducible representation Y is an eigenstate of this operator for the eigenvalue

$$s_Y = \sum_{l} \frac{\lambda_l(\lambda_l - 1)}{2} - \sum_{c} \frac{\bar{\lambda}_c(\bar{\lambda}_c - 1)}{2} \tag{16}$$

where $\lambda_l(\bar{\lambda}_c)$ is the number of cases in the lth line (cth column) of the Young pattern Y. As a result, the harmonic basis (8) labelled by n_k, Y, a and b is also an eigenstate basis of the preceding Hamiltonian (15) but for the energy spectrum

$$E = \sum_{k=1}^{N} k n_k \omega + \tfrac{1}{2}N\omega + d_{Y,a}\omega + s_Y \nu\omega \tag{17}$$

with $d_{Y,a}$ unaltered. For a given Y, this spectrum reproduces the harmonic spectrum of the representation Y at $\nu = 0$ and, surprising enough, it reproduces the harmonic spectrum of the conjugate representation $\bar{Y}$ at $\nu = 1$. Indeed, the relation $d_{Y,a} + s_Y = d_{\bar{Y},a}$ is clearly verified by the values of $d_{Y,a}$ obtained above for $N = 2, 3, 4, \ldots$ (the proof for all N remains to be obtained). In conclusion, the eigenstates are those of a system of identical harmonic oscillators but the spectral properties are those of a model of intermediate statistics.

The partition functions are simply expressed in terms of the harmonic ones (11) as

$$Z_Y(\nu) = \mathrm{e}^{-s_Y\beta\omega\nu} Z_Y(0) \tag{18}$$

and the class functions are then deduced from the relations (1). The linear interpolation between conjugate representations means that $Z_Y(1-\nu) = Z_{\bar{Y}}(\nu)$ which is equivalent to

$Z_P(1-\nu) = (-)^P Z_P(\nu)$ due to the identity $\chi_{\bar{Y}}(P) = (-)^P \chi_Y(P)$. The class functions have other remarkable properties that I have verified up to $N = 6$ (their proofs for all N remain to be obtained). After some factorizations, one recovers the functions $F_P(\nu) = Z_P(\nu)/Z_P(0)$ obtained in [3] for a different model (this coincidence will be discussed later). The appendix displays the first functions F_P and their connected parts. The cyclic functions are completely factorizable as

$$F_{[N]} = \prod_{k=1}^{N-1} \frac{\mathrm{sh}(k - N\nu)(\beta\omega/2)}{\mathrm{sh}(k\beta\omega/2)}. \tag{19}$$

Albeit the function F_P is not completely factorizable in the general case, it admits the same zeros as $\Pi_L F_{[L]}^{\nu_L}$, namely $\nu = 1/L, 2/L, \ldots, (L-1)/L$ with the multiplicity ν_L for each L. These zeros are present owing to certain coincidences between the spectra of distinct representations.

At last, let us compute the connected part (4) of the class functions in the thermodynamic limit $\omega \to 0$ with the appropriate prescription for a 1D space. Doing this up to $N = 6$ for instance, one verifies the polynomials

$$Z^{\mathrm{c}}_{[N]} = \frac{V}{\lambda_T} \frac{1}{\sqrt{N}} \prod_{k=1}^{N-1} \left(1 - \frac{N}{k}\nu\right) \qquad Z^{\mathrm{c}}_{P \neq [N]} = 0. \tag{20}$$

This reproduces the cluster coefficients $b_N = (\pm)^{N-1} N^{-1} Z^{\mathrm{c}}_{[N]}$ for Bose (Fermi) statistics, because the spectrum of the Calogero model in a harmonic well is linear with ν and thus coincides with its first-order approximation. On the other hand, one has merely $b_{N>2} = 0$ for Boltzmann statistics. These results are supported by a perturbative analysis. Indeed, Z^{c}_P can be expanded as a series of connected cluster diagrams with ν_L loops of length L according to [3]. Consider a connected diagram with n vertices and ℓ loops. To compute it, one has to effect the interchange of the two incoming propagators in each vertex in order to reduce the interaction $\nu\omega p_{ij}$ to a constant $\nu\omega$. These interchanges produce a new topology with ℓ' loops. Integrating the harmonic propagators, each loop gives one one-body partition function for a certain temperature, and thus the diagram behaves as $\omega^{n-\ell'}$ in the limit $\omega \to 0$. Since the maximal number of loops is $\ell' = n + 1$ in a connected diagram with n vertices, the leading diagrams have $\ell' = n + 1$ loops after the interchanges and one verifies that such diagrams have $\ell = 1$ loop in their initial form [3]. These diagrams behave as ω^{-1} which has to be identified with a volume, and the other diagrams do not contribute. This proves that only $Z^{\mathrm{c}}_{[N]}$ contributes, the other class functions vanish in the thermodynamic limit.

We have obtained the following thermodynamics. In Bose and Fermi statistics, the linear model reproduces the thermodynamics of the Calogero model where it is well known that particles of the same momentum obey the Haldane exclusion of parameter $g = \nu$ for bosons and $g = -\nu$ for fermions [4]. In contrast, in the academic case of Boltzmann statistics, the dependence on the coupling parameter disappears and thus the linear model verifies the state equation of an ideal gas.

We are now in position to construct a 2D extension for our linear model of intermediate statistics in a harmonic well. This extension is suggested in [3], where one has obtained a few-body spectrum which interpolates linearly between the Bose spectrum and the Fermi spectrum of a system of independent and identical 2D harmonic oscillators and which is consistent with a finite virial expansion. Indeed, the 2D functions $F_P(\nu) = Z_P(\nu)/Z_P(0)$ obtained in [3] are identical with those of the 1D linear model of intermediate statistics, and this indicates that the 2D linear model is merely the tensorial product of the 1D linear model by a system of 1D harmonic oscillators. The Hamiltonian is then of the form

$$H = \sum_{i=1}^{N}(-\tfrac{1}{2}\partial_{x_i}^2 - \tfrac{1}{2}\partial_{y_i}^2 + \tfrac{1}{2}\omega^2 x_i^2 + \tfrac{1}{2}\omega^2 y_i^2) + \sum_{i,j<} \nu\omega p_{x_i x_j} \tag{21}$$

where $\boldsymbol{r}_i = (x_i, y_i)$ is a couple of coordinates for the ith particle on the plane. The tensorial product gives an eigenstate basis in terms of the eigenstates of a system of 1D harmonic oscillators,

$$\psi = \langle x_i | n_k, Y, a, b\rangle \langle y_i | n'_k, Y', a', b'\rangle \tag{22}$$

with the energy spectrum

$$E = \sum_{k=1}^{N} k n_k \omega + \sum_{k=1}^{N} k n'_k \omega + N\omega + (d_{Y,a} + d_{Y',a'})\omega + s_Y \nu\omega. \tag{23}$$

However, these eigenstates have to be symmetrized according to the irreducible representations (Bose, Fermi, mixed) of the group of the particle exchanges.

Knowing the F_P functions from the 1D linear model, we directly have the class functions for the 2D linear model as

$$Z_P = F_P \prod_L \left(\frac{1}{2\mathrm{sh}(L\beta\omega/2)}\right)^{2\nu_L} \tag{24}$$

and we can deduce the partition functions from the relations (2). The property $F_P(1-\nu) = (-)^P F_P(\nu)$ still implies $Z_Y(1-\nu) = Z_{\bar{Y}}(\nu)$ due to the identity $\chi_{\bar{Y}}(P) = (-)^P \chi_Y(P)$, and thus the spectral properties of the 2D linear model are those of an intermediate statistics interpolating linearly between conjugate representations of S_N when ν goes from 0 to 1.

Computing the thermodynamic limit of the connected part of the class functions up to $N = 6$, one verifies the following polynomials in ν [12],

$$Z_P^{\mathrm{c}} = \frac{V}{\lambda_T^2} N^{\sum \nu_L - 2}\{\nu(\nu-1)\}^{\sum \nu_L - 1} \prod_L F_{[L]}^{\nu_L} \tag{25}$$

where

$$F_{[L]} = \prod_{k=1}^{L-1}\left(1 - \frac{L}{k}\nu\right). \tag{26}$$

These formulae can be proved for all N by means of a perturbative analysis provided that some zeros of the F_P's are known. The perturbative expansion of Z_P^{c} involves the above mentioned diagrams for the 1D linear model, but these diagrams are now multiplied by the contribution (12) of the second dimension y and, furthermore, the thermodynamic limit is different for a 2D space. Only the main steps of the reasoning are outlined here.

(i) By definition, the diagrams of Z_P^{c} are connected and have $\ell = \sum_L \nu_L$ loops. The topological inequality $\ell + \ell' \leqslant n + 2$ ensures that the leading diagrams behave as a volume factor ω^{-2} in the limit $\omega \to 0$.

(ii) As a result, the leading diagrams verify $n = \ell + \ell' - 2 \leqslant \ell + N - 2$ so that Z_P^{c} is a polynomial of degree $\ell + N - 2$ at most.

(iii) At least $n = \ell - 1$ vertices are required to connect all the loops and thus the polynomial begins as $\nu^{\ell-1}$. In other words, the multiplicity of the zero $\nu = 0$ is $\ell - 1$. Due to the symmetry of the spectrum under the mirror transformation $\nu \to 1 - \nu$, the value $\nu = 1$ is also a zero of Z_P^{c} with the same multiplicity.

(iv) Assuming that Z_P admits the same zeros that $\Pi_L F_{[L]}^{\nu_L}$ for all ω, the relations (4) imply obviously that Z_P^{c} also have these zeros with the same multiplicities.

(v) At this step, one has obtained $\ell + N - 2$ zeros for a polynomial Z_P^{c} whose maximal degree is precisely $\ell+N-2$. Thus, the polynomial expression (25) is proved up to a constant factor. To determine this constant factor, it suffices to compute the lowest coefficient in the polynomial. One has to sum the diagrams with $n = \ell - 1$ vertices. In fact, one shows that this sum is determined by the total multiplicity $n!N^{n-1}\Pi_L L^{\nu_L}$ of these diagrams, and one finally reproduces (25).

On the other hand, up to $N = 6$ for instance, one easily verifies the following formula for the cluster coefficients in Bose (Fermi) statistics:

$$b_N = \sum_P \frac{d_P}{N!}(\pm)^P Z_P^{\mathrm{c}} = (\pm)^{N-1}\frac{V}{\lambda_T^2}\frac{1}{N^2}\prod_{k=1}^{N-1}\left(1 \mp \frac{N}{k}\nu(1 \pm 1 - \nu)\right). \qquad (27)$$

These coefficients coincide with those of a system of bosons (fermions) obeying Haldane statistics of the parameter $\nu(1 \pm 1 - \nu)$ [3]. The corresponding thermodynamics has been studied [13]. In particular, one shows that the second virial coefficient is identical with the anyon one and the others do not depend on the coupling parameter ν. Note that the polynomials (25) and (26) play an important role in the perturbative and numerical approaches to the anyon thermodynamics [3, 6, 7, 14]. Indeed, the results for $Z_{[11]}^{\mathrm{c}}$, $Z_{[21]}^{\mathrm{c}}$ and $Z_{[N]}^{\mathrm{c}}$ are exact in the anyon context whereas the other polynomials may be viewed as a starting approximation whose anyonic corrections remain to be elucidated.

Let me also mention the academic case of Boltzmann statistics. The cluster coefficients are then

$$b_N = \frac{1}{N!}Z_{[1^N]}^{\mathrm{c}} = V\lambda_T^{-2}N^{N-2}N!^{-1}\{\nu(\nu-1)\}^{N-1}.$$

In fact these coefficients are reproduced in a gas obeying Haldane statistics of the parameter $g = \nu(1-\nu)$ between particles of the same momentum. One easily derives the thermodynamics in a closed form for such a gas by maximizing the thermodynamical potential, so then the cluster expansion is deduced by solving an implicit equation by iterations. The pressure is $k_{\mathrm{B}}T\rho + \frac{1}{2}\nu(1-\nu)k_{\mathrm{B}}T\lambda_T^2\rho^2$ in terms of the density ρ.

The existence of well defined thermodynamics connected to anyons is not at all obvious for a Hamiltonian of the form (21). In the absence of a complete understanding, only a few aspects of the connection between the anyon model and the 2D linear model are displayed here.

The first aspect regards the presence of the anyon linear energies in the 2D linear model. For anyons in the irreducible representation Y, the two classes of linear energies are [3]

$$E_{\mathrm{I}} = \sum_{n=0}^{1}\sum_{m=1-n}^{N-n}(n+m)\lambda_{nm}\omega + N\omega + d_{Y,a}\omega + \tfrac{1}{2}N(N-1)\nu\omega \qquad (28)$$

$$E_{\mathrm{II}} = \sum_{n=0}^{1}\sum_{m=1-n}^{N-n}(n+m)\lambda_{nm}\omega + N\omega + d_{\bar{Y},a}\omega + \tfrac{1}{2}N(N-1)(1-\nu)\omega$$

where the quantum numbers λ_{nm} are non-negative integers. In the 2D linear model, the energies with the same dependence in ν correspond respectively to the symmetric and antisymmetric eigenfunctions under the exchanges of the x_i's. The tensorial product (22) leads directly to a basis of these eigenstates in the representation Y, namely

$$\psi_{\mathrm{I}} = \langle x_i|n_k, \mathrm{Bose}\rangle\langle y_i|n_k', Y, a, b\rangle \qquad (29)$$

$$\psi_{\mathrm{II}} = \langle x_i|n_k, \mathrm{Fermi}\rangle\langle y_i|n_k', \bar{Y}, a, b\rangle.$$

Albeit these eigenfunctions have no connection with the anyon ones, their energies (23) exactly coincide with the anyon linear energies (28).

We would like to connect the interaction of the 2D linear model to the anyon interaction at first perturbative order. The complex notations $z_i = x_i + \mathrm{i}y_i$ and $\partial_i = \partial_{z_i}$ are used here. Starting from the definition of a system of N anyons in the anyon gauge with an additional harmonic attraction, it is advisable to do the transformation $\psi = \Pi_{i,j<} z_{ij}^{\nu} \tilde{\psi}$ in order to work with both a monovalued wavefunction $\tilde{\psi}$ and a Hamiltonian

$$\tilde{H} = \sum_{i=1}^{N} (-\tfrac{1}{2}\Delta_i + \tfrac{1}{2}\omega^2 r_i^2) - 2\nu \sum_{i,j<} \frac{1}{z_i - z_j} (\bar{\partial}_i - \bar{\partial}_j) \tag{30}$$

whose matrix elements are finite. This last formulation is suitable to a perturbative analysis [15]. At first perturbative order, it suffices to diagonalize the restriction of the perturbation inside each unperturbed degenerate subspace. The restriction of $-2\nu z_{ij}^{-1} \bar{\partial}_{ij}$ naturally appears in the calculation of the 2-anyon spectrum at first perturbative order. It is given by $\nu\omega h_{ij}$ where h_{ij} is the helicity operator whose eigenvalue is the sign of the relative angular momentum m_{ij} of the two particles i and j with sign(0) = 1 (sign(0) = −1 would correspond to a self-adjoint extension of the anyon model). At this level, the difference between the anyon model and the linear model consists of a simple rotation

$$h_{ij} = U_{ij}^{-1} p_{x_i x_j} U_{ij} \qquad U_{ij} = (U_{ij}^{+})^{-1} = \frac{1}{\sqrt{2}} (1 + p_{x_i x_j} h_{ij}) \tag{31}$$

due to the elementary relations $h_{ij}^2 = p_{ij}^2 = 1$, $h_{ij}^{+} = h_{ij}$, $p_{ij}^{+} = p_{ij}$ and $h_{ij} p_{ij} + p_{ij} h_{ij} = 0$ which can easily be verified onto the angular basis $\mathrm{e}^{\mathrm{i}m_{ij} \arg(\boldsymbol{r}_{ij})}$ with m_{ij} an integer, except for the subspace $m_{ij} = 0$ where U_{ij} must be set to unity. In the N-body case, the relation

$$\sum_{i,j<} \nu\omega h_{ij} = \sum_{i,j<} U_{ij}^{-1} \nu\omega p_{x_i x_j} U_{ij} \tag{32}$$

implies the identity of the traces of the energies inside each unperturbed degenerate subspace between the 2D linear model and the anyon model at first perturbative order in ν. The identity also holds for the partition and class functions at first order in ν. Without a relationship between the linear model and the anyon model at further orders in ν, the finiteness of the virial coefficients of the linear model in the thermodynamic limit and their relevance in the anyon context remains mysterious. This should be contrasted with the linearization of the ν-dependence in the three-anyon spectrum which leads to a divergent virial coefficient as it should be [16].

In conclusion, we have solved a strange model of intermediate statistics and we have obtained some of its physical and mathematical implications. However, the definition of the linear model as a first-order approximation in a harmonic well is not satisfactory. For example, the use of a box instead of a harmonic well does not lead to the correct virial expansion without introducing nonlinear statistics-dependent terms in the Hamiltonian [17]. In fact, a satisfying construction, if it is possible, remains to be stated. One would like to construct a physical model defined independently of the regulator (box, harmonic well, etc) and whose eigenstates also interpolate between bosons and fermions. In the absence of such a construction, the physical interpretation of the results presented in this paper seems out of reach. However, two points deserve attention. In the 1D linear model, the two-body interaction seems to mimic the particle interchange observed in the asymptotic scattering of the Calogero problem [18]. In the 2D linear model, the interaction operator $p_{x_i x_j}$ is nothing but the parity in the relative framework, that is $p_{x_i x_j}(x_{ij}, y_{ij}) = (-x_{ij}, y_{ij})$. I hope that a different point of view or an improvement in these models will shed some light on the connection between anyons and the generalized exclusion principle.

Appendix

The calculation of the first class and partition functions can easily be performed on a formal computer by handling rational functions in the variables $u = \mathrm{e}^{\beta\omega}$, $v = \mathrm{e}^{\beta\omega\nu}$. It appears that the cyclic functions $F_{[N]}$ are completely factorizable according to (19). As an illustration, the other functions F_P are presented here in their factorized form up to $N = 4$,

$$
\begin{aligned}
F_{[11]} &= \frac{\mathrm{ch}(1-2\nu)(\beta\omega/2)}{\mathrm{ch}(\beta\omega/2)} \\
F_{[111]} &= \frac{\mathrm{ch}(3-6\nu)(\beta\omega/2) + 2\mathrm{ch}(\beta\omega/2)}{\mathrm{ch}(\beta\omega/2)(2\mathrm{ch}\beta\omega+1)}, \\
F_{[1^4]} &= \frac{\mathrm{ch}(3-6\nu)\beta\omega + 3(2\mathrm{ch}\beta\omega+1)\mathrm{ch}(1-2\nu)\beta\omega + 2\mathrm{ch}\beta\omega}{4\mathrm{ch}^2(\beta\omega/2)\mathrm{ch}\beta\omega(2\mathrm{ch}\beta\omega+1)} \qquad (33) \\
F_{[211]} &= \frac{\mathrm{ch}(1-2\nu)(\beta\omega/2)(\mathrm{ch}(2-4\nu)\beta\omega + 2\mathrm{ch}^2(\beta\omega/2))}{\mathrm{ch}(\beta\omega/2)\mathrm{ch}\beta\omega(2\mathrm{ch}\beta\omega+1)} F_{[2]} \\
F_{[22]} &= \frac{\mathrm{ch}(2-4\nu)\beta\omega + 2\mathrm{ch}(1-2\nu)\beta\omega - 2\mathrm{sh}^2(\beta\omega/2)}{\mathrm{ch}\beta\omega(2\mathrm{ch}\beta\omega+1)} F_{[2]}^2 \qquad (34)
\end{aligned}
$$

$F_{[21]} = F_{[2]}(3\beta)$ where $F_{[2]}(\beta)$ is a factor, and $F_{[31]} = F_{[3]}(2\beta)$ where $F_{[3]}(\beta)$ is a factor in the same way. The connected parts $F_P^{\mathrm{c}}(\nu) = Z_P^{\mathrm{c}}(\nu)/Z_P(0)$ are easily deduced. Indeed, due to its product form (12), the normalization $Z_P(0)$ can be factorized out of the relations (4) so that these relations also hold between the F_P's and the F_P^{c}'s. Using the notation

$$
G(\beta) = 4\mathrm{sh}\nu(\beta\omega/2)\mathrm{sh}(\nu-1)(\beta\omega/2) \qquad (35)
$$

the connected parts in a factorized form read

$$
\begin{aligned}
F_{[11]}^{\mathrm{c}} &= \frac{G}{2\mathrm{ch}(\beta\omega/2)} \\
F_{[111]}^{\mathrm{c}} &= \frac{\mathrm{ch}(1-2\nu)(\beta\omega/2) + 2\mathrm{ch}(\beta\omega/2)}{\mathrm{ch}(\beta\omega/2)(2\mathrm{ch}\beta\omega+1)} G^2 \\
F_{[21]}^{\mathrm{c}} &= \frac{G(2\beta)}{2\mathrm{ch}\beta\omega+1} F_{[2]} \\
F_{[1^4]}^{\mathrm{c}} &= \frac{(\mathrm{ch}(1-2\nu)(\beta\omega/2) + \mathrm{ch}(\beta\omega/2))^3 + \frac{1}{2}\mathrm{ch}\beta\omega(3\mathrm{ch}(1-2\nu)(\beta\omega/2) + 5\mathrm{ch}(\beta\omega/2))}{\mathrm{ch}^2(\beta\omega/2)\mathrm{ch}\beta\omega(2\mathrm{ch}\beta\omega+1)} G^3 \qquad (36) \\
F_{[211]}^{\mathrm{c}} &= \frac{\mathrm{ch}(1-2\nu)\beta\omega + 2\mathrm{ch}(1-2\nu)(\beta\omega/2)\mathrm{ch}(\beta\omega/2) + 2\mathrm{ch}\beta\omega + 1}{2\mathrm{ch}(\beta\omega/2)\mathrm{ch}\beta\omega(2\mathrm{ch}\beta\omega+1)} G(\beta)G(2\beta)F_{[2]} \\
F_{[22]}^{\mathrm{c}} &= \frac{\mathrm{ch}(1-2\nu)\beta\omega + 2\mathrm{ch}^2(\beta\omega/2)}{\mathrm{ch}\beta\omega(2\mathrm{ch}\beta\omega+1)} G(2\beta)F_{[2]}^2 \\
F_{[31]}^{\mathrm{c}} &= \frac{G(3\beta)}{4\mathrm{ch}(\beta\omega/2)\mathrm{ch}\beta\omega} F_{[3]}
\end{aligned}
$$

and $F_{[N]}^{\mathrm{c}} = F_{[N]}$. Performing the connected part, the factor $\Pi_L F_{[L]}^{\nu_L}$ is maintained whereas a new factor $G^{\sum \nu_L - 1}$ can always be extracted.

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